

# A NEW CLASS OF SUPERMATRIX ALGEBRAS DEFINED BY TRANSITIVE MATRICES

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**ABSTRACT.** We give a natural definition for the transitivity of a matrix. Using an endomorphism  $\delta : R \rightarrow R$  of a base ring  $R$  and a transitive  $n \times n$  matrix  $T \in M_n(Z(R))$  over the center  $Z(R)$ , we construct the subalgebra  $M_n(R, \delta, T)$  of the full  $n \times n$  matrix algebra  $M_n(R)$  consisting of the so called  $n \times n$  supermatrices. If  $\delta^n = \text{id}_R$  and  $T$  satisfies some extra conditions, then we exhibit an embedding  $\bar{\delta} : R \rightarrow M_n(R, \delta, T)$ . An other result is that  $M_n(R, \delta, T)$  is closed with respect to taking the (pre)adjoint. If  $R$  is Lie nilpotent and  $A \in M_n(R, \delta, T)$ , then the use of the preadjoint and the corresponding determinants and characteristic polynomials yields a Cayley-Hamilton identity for  $A$  with right coefficients in the fixed ring  $\text{Fix}(\delta)$ . The presence of a primitive  $n$ -th root of unity and  $\delta^n = \text{id}_R$  guarantee the right integrality of a Lie nilpotent  $R$  over  $\text{Fix}(\delta)$ . We present essentially new supermatrix algebras over the Grassmann algebra.

## 1. INTRODUCTION

Throughout the paper a ring  $R$  means a not necessarily commutative ring with identity, all subrings inherit and all endomorphisms preserve the identity. The group of units in  $R$  is denoted by  $U(R)$  and the centre of  $R$  is denoted by  $Z(R)$ .

In Section 2 we give a natural definition for the transitivity of an  $n \times n$  matrix over  $R$ . A complete description of such transitive matrices is provided. The "blow-up" construction gives an easy way to build bigger transitive matrices starting from a given one.

In Section 3 we prove that the Hadamard multiplication by a transitive matrix  $T \in M_n(Z(R))$  gives a new type of automorphisms of the full  $n \times n$  matrix algebra  $M_n(R)$ . We use an automorphism of the above type and an endomorphism  $\delta_n$  of  $M_n(R)$  naturally induced by an endomorphism  $\delta : R \rightarrow R$ , to define the subalgebra  $M_n(R, \delta, T)$  of  $M_n(R)$  consisting of the so called  $n \times n$  supermatrices. If  $\delta^n = \text{id}_R$  and  $T$  satisfies some extra conditions, then we exhibit an embedding  $\bar{\delta} : R \rightarrow M_n(R, \delta, T)$  of  $R$  into the  $n \times n$  supermatrix algebra  $M_n(R, \delta, T)$ .

Section 4 is devoted to the study of the right and left (and symmetric) determinants and the corresponding right and left (and symmetric) characteristic

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polynomials of supermatrices. The main result of Section 4 claims that the supermatrix algebra  $M_n(R, \delta, T)$  is closed with respect to taking the preadjoint. As a consequence, we obtain that the mentioned determinants and the coefficients of the corresponding characteristic polynomials are in the fixed ring  $\text{Fix}(\delta)$  of  $\delta$ . If  $R$  is Lie nilpotent of index  $k$ , then we derive that any supermatrix  $A \in M_n(R, \delta, T)$  satisfies a Cayley-Hamilton identity (of degree  $n^k$ ) with right coefficients in  $\text{Fix}(\delta)$ . If  $K$  is a field of characteristic zero and there is a primitive  $n$ -th root of unity in  $K$ , then a combination of the embedding result in Section 3 and the Cayley-Hamilton identity in Section 4 gives that a Lie nilpotent  $K$ -algebra  $R$  is right (and left) integral over  $\text{Fix}(\delta)$ , where  $\delta : R \rightarrow R$  is a  $K$ -automorphism with  $\delta^n = \text{id}_R$ .

In Section 5 we explain in detail how the results of Section 4 generalize the earlier results in [S2]. In order to demonstrate the importance of supermatrices, we mention their role in Kemer's theory of T-ideals (see [K]). New examples of supermatrix algebras over the Grassmann algebra are presented in 5.2 and 5.3. These examples show that our supermatrix algebras introduced in Section 3 form a rich class of algebras.

## 2. TRANSITIVE MATRICES

Let  $M_n(R)$  denote the ring of  $n \times n$  matrices over a (not necessarily commutative) ring  $R$  with 1. A matrix  $T = [t_{i,j}]$  in  $M_n(R)$  is called *transitive* if

$$t_{i,i} = 1 \text{ and } t_{i,j}t_{j,k} = t_{i,k} \text{ for all } i, j, k \in \{1, \dots, n\}.$$

Notice that  $t_{i,j}t_{j,i} = t_{i,i} = 1$  and  $t_{j,i}t_{i,j} = t_{j,j} = 1$  imply that  $t_{i,j}$  and  $t_{j,i}$  are (multiplicative) inverses of each other.

**2.1. Proposition.** *For a matrix  $T \in M_n(R)$  the following are equivalent.*

- (1)  *$T$  is transitive.*
- (2) *There exists a sequence  $g_i \in U(R)$ ,  $1 \leq i \leq n$  of invertible elements such that  $t_{i,j} = g_i g_j^{-1}$  for all  $i, j \in \{1, \dots, n\}$ . If  $h_i \in U(R)$ ,  $1 \leq i \leq n$  is another sequence with  $t_{i,j} = h_i h_j^{-1}$ , then  $h_i = g_i c$  for some constant  $c \in U(R)$ .*

- (3) *There exists an  $n \times 1$  (column) matrix  $G = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$  with invertible entries*

*$g_i \in U(R)$ ,  $1 \leq i \leq n$  such that  $T = G\tilde{G}$ , where  $\tilde{G} = [g_1^{-1}, \dots, g_n^{-1}]$  is a  $1 \times n$  (row) matrix.*

**Proof.** (1) $\implies$ (2): Take  $g_i = t_{i,1}$ , then the transitivity of  $T$  ensures that  $t_{i,j} = t_{i,1}t_{1,j} = t_{i,1}t_{j,1}^{-1} = g_i g_j^{-1}$ . Clearly,  $t_{i,1} = g_i g_1^{-1} = h_i h_1^{-1}$  implies that  $h_i = g_i c$ , where  $c = g_1^{-1}h_1$ .

(2) $\implies$ (1): Now  $t_{i,i} = g_i g_i^{-1} = 1$  and  $t_{i,j}t_{j,k} = g_i g_j^{-1} g_j g_k^{-1} = g_i g_k^{-1} = t_{i,k}$ .

(2) $\iff$ (3): Obvious.  $\square$

The *Hadamard product* of the matrices  $A = [a_{i,j}]$  and  $B = [b_{i,j}]$  in  $M_n(R)$  is defined as  $A * B = [a_{i,j}b_{i,j}]$ .

**2.2. Proposition.** *If  $T = [t_{i,j}]$  is a transitive matrix in  $M_n(R)$ , then  $T^2 = nT$ . If  $R$  is commutative and  $S \in M_n(R)$  is another transitive matrix, then  $T * S$  is also transitive.*

**Proof.** The  $(i, j)$  entry of the square  $T^2$  is

$$\sum_{k=1}^n t_{i,k} t_{k,j} = \sum_{k=1}^n t_{i,j} = n t_{i,j}.$$

For a commutative  $R$ , the transitivity of the Hadamard product  $T * S$  is obvious.  $\square$

**2.3. Proposition ("blow up").** For a transitive matrix  $T = [t_{i,j}]$  in  $M_n(R)$  and for a sequence  $0 = d_0 < d_1 < \dots < d_{n-1} < d_n = m$  of integers define an  $m \times m$  matrix  $\hat{T} = [\hat{t}_{p,q}]$  (the blow up of  $T$ ) as follows:

$$\hat{t}_{p,q} = t_{i,j} \text{ if } d_{i-1} < p \leq d_i \text{ and } d_{j-1} < q \leq d_j.$$

The above  $\hat{T}$  is a transitive matrix in  $M_m(R)$ . If necessary, we use the notation  $T(d_1, \dots, d_{n-1}, d_n)$  instead of  $\hat{T}$ .

**Proof.**  $\hat{T} = [T_{i,j}]$  can be considered as an  $n \times n$  matrix of blocks, the size of the block  $T_{i,j}$  in the  $(i, j)$  position is  $(d_i - d_{i-1}) \times (d_j - d_{j-1})$  and each entry of  $T_{i,j}$  is  $t_{i,j}$ . The integers  $p, q, r \in \{1, \dots, m\}$  uniquely determine the indices  $i, j, k \in \{1, \dots, n\}$  satisfying  $d_{i-1} < p \leq d_i$ ,  $d_{j-1} < q \leq d_j$  and  $d_{k-1} < r \leq d_k$ . The definition of  $\hat{T} = [\hat{t}_{p,q}]$  and the transitivity of  $T$  ensure that

$$\hat{t}_{p,q} \hat{t}_{q,r} = t_{i,j} t_{j,k} = t_{i,k} = \hat{t}_{p,r}.$$

Thus the  $m \times m$  matrix  $\hat{T}$  is also transitive.  $\square$

**2.4. Examples.** For an invertible element  $u \in U(R)$ , the sequence  $g_i = u^{i-1}$ ,  $1 \leq i \leq n$  in Proposition 2.1 gives an  $n \times n$  matrix  $P^{(u)} = [p_{i,j}]$  with  $p_{i,j} = u^{i-j}$ . The choice  $u = 1$  yields the Hadamard identity  $H_n$  (each entry of  $H_n$  is 1). If  $n = 2$  and  $u = -1$ , then we obtain the following  $2 \times 2$  matrix

$$P^{(-1)} = P = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For  $d_1 = d$  and  $d_2 = m$ , the blow up

$$\hat{P} = P(d, m) = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix},$$

of  $P$  (see Proposition 2.3) contains the square blocks  $P_{1,1}$  and  $P_{2,2}$  of sizes  $d \times d$  and  $(m-d) \times (m-d)$  and the rectangular blocks  $P_{1,2}$  and  $P_{2,1}$  of sizes  $d \times (m-d)$  and  $(m-d) \times d$ . Each entry of  $P_{1,1}$  and  $P_{2,2}$  is 1 and each entry of  $P_{1,2}$  and  $P_{2,1}$  is  $-1$ . Thus  $P^{(u)}$ ,  $H_n$ ,  $P$  and  $P(d, m)$  are examples of transitive matrices.  $\square$

### 3. THE ALGEBRA OF SUPERMATRICES

In the present section we consider certain endomorphisms of the full matrix algebra  $M_n(R)$ . A typical example is the conjugate automorphism  $X \mapsto W^{-1} X W$ , where  $W \in GL_n(R)$  is an invertible matrix (see the well known Skolem-Noether theorem). Any endomorphism  $\delta : R \rightarrow R$  of  $R$  can be naturally extended to an endomorphism  $\delta_n : M_n(R) \rightarrow M_n(R)$ . The following proposition provides a further type of automorphisms of  $M_n(R)$ .

**3.1. Proposition.** Let  $T = [t_{i,j}]$  be a matrix in  $M_n(Z(R))$ , where  $Z(R)$  denotes the center of  $R$ . The following conditions are equivalent:

- (1)  $T$  is transitive,  
 (2) for  $A \in M_n(R)$  the map (Hadamard multiplication by  $T$ )  $\Theta_T(A) = T * A$  is an automorphism of the matrix algebra  $M_n(R)$  (over  $Z(R)$ ).

**Proof.** (1) $\implies$ (2): In order to prove the multiplicative property of  $\Theta_T$ , it is enough to check that  $T * (AB) = (T * A)(T * B)$  for all  $A, B \in M_n(R)$ . Indeed, the  $(i, j)$  entries of  $(T * A)(T * B)$  and  $T * (AB)$  are equal:

$$\sum_{k=1}^n t_{i,k} a_{i,k} t_{k,j} b_{k,j} = \sum_{k=1}^n t_{i,k} t_{k,j} a_{i,k} b_{k,j} = \sum_{k=1}^n t_{i,j} a_{i,k} b_{k,j} = t_{i,j} \sum_{k=1}^n a_{i,k} b_{k,j}.$$

The inverse of  $\Theta_T$  is  $\Theta_T^{-1}(A) = S * A$ , where  $S = [t_{i,j}^{-1}]$  is also transitive in  $M_n(Z(R))$ .

(2) $\implies$ (1): Now  $t_{i,i} = 1$  is a consequence of  $T * I_n = I_n$ . Using the standard matrix units  $E_{i,j}$  and  $E_{j,k}$  in  $M_n(R)$ , the multiplicative property of  $\Theta_T$  gives that

$$\begin{aligned} t_{i,k} E_{i,k} &= T * E_{i,k} = T * (E_{i,j} E_{j,k}) = (T * E_{i,j})(T * E_{j,k}) \\ &= (t_{i,j} E_{i,j})(t_{j,k} E_{j,k}) = t_{i,j} t_{j,k} E_{i,k}, \end{aligned}$$

whence  $t_{i,k} = t_{i,j} t_{j,k}$  follows.  $\square$

If  $\Delta, \Theta : S \longrightarrow S$  are endomorphisms of the ring  $S$ , then the subset

$$S(\Delta = \Theta) = \{s \in S \mid \Delta(s) = \Theta(s)\}$$

is a subring of  $S$  (notice that  $\Delta(1) = \Theta(1) = 1$ ). Let  $\text{Fix}(\Delta) = S(\Delta = \text{id}_S)$  denote the subring of the fixed elements of  $\Delta$ .

Now take  $S = M_n(R)$ ,  $\Delta = \delta_n$  and  $\Theta_T(A) = T * A$ , where  $\delta : R \longrightarrow R$  is an endomorphism and  $T = [t_{i,j}]$  is a transitive matrix in  $M_n(Z(R))$ . The short notation for  $M_n(R)(\delta_n = \Theta_T)$  is

$$M_n(R, \delta, T) = \{A \in M_n(R) \mid A = [a_{i,j}] \text{ and } \delta(a_{i,j}) = t_{i,j} a_{i,j} \text{ for all } 1 \leq i, j \leq n\}.$$

If  $C \subseteq Z(R) \cap \text{Fix}(\delta)$  is a (commutative) subring (say  $C = \mathbb{Z}$ ), then  $M_n(R, \delta, T)$  is a  $C$ -subalgebra of  $M_n(R)$ . The elements of the *supermatrix algebra*  $M_n(R, \delta, T)$  are called  $(\delta, T)$ -*supermatrices*. If  $t_{i,j} \in \text{Fix}(\delta)$  for all  $1 \leq i, j \leq n$ , then  $M_n(R, \delta, T)$  is closed with respect to the action of  $\delta_n$ .

**3.2. Theorem.** Let  $T = [t_{i,j}]$  be an  $n \times n$  transitive matrix such that the entries  $t_{i,1} \in U(R) \cap Z(R)$ ,  $1 \leq i \leq n$  of the first column are central invertible elements. If  $\frac{1}{n} \in R$  and  $\delta : R \longrightarrow R$  is an arbitrary endomorphism, then for  $r \in R$  take  $\bar{\delta}(r) = \frac{1}{n} [x_{i,j}(r)]_{n \times n}$ , where

$$x_{i,j}(r) = \sum_{k=0}^{n-1} t_{j,i}^k \delta^k(r) = \sum_{k=0}^{n-1} t_{i,j}^{-k} \delta^k(r)$$

and  $\frac{1}{n} x_{i,j}(r)$  is in the  $(i, j)$  position of the  $n \times n$  matrix  $\bar{\delta}(r)$ . The above definition gives a map  $\bar{\delta} : R \longrightarrow M_n(R)$  such that  $\bar{\delta}(cr) = c\bar{\delta}(r)$  and  $\bar{\delta}(rc) = \bar{\delta}(r)c$  for all  $c \in \text{Fix}(\delta)$ .

- (1) If  $t_{i,1}^n = 1$  and  $1 - t_{i,j}$  is a non-zero divisor in  $Z(R)$  for all  $1 \leq i, j \leq n$  with  $i \neq j$ , then  $\bar{\delta}(r) = rI_n$  is a scalar matrix for any  $r \in \text{Fix}(\delta)$ .  
 (2) If  $t_{1,1}^k + t_{2,1}^k + \cdots + t_{n,1}^k = t_{1,1}^{-k} + t_{2,1}^{-k} + \cdots + t_{n,1}^{-k} = 0$  for all  $1 \leq k \leq n-1$ , then  $\bar{\delta} : R \longrightarrow M_n(R)$  is an embedding of rings.  
 (3) If  $t_{i,1} \in \text{Fix}(\delta)$ ,  $t_{i,1}^n = 1$  for all  $1 \leq i \leq n$  and  $\delta^n = \text{id}_R$ , then  $\bar{\delta}$  is an  $R \longrightarrow M_n(R, \delta, T)$  embedding.

**Proof.** Clearly,  $t_{i,j} = t_{i,1}t_{j,1}^{-1}$  ensures that  $t_{i,j} \in U(R) \cap Z(R)$  for all  $1 \leq i, j \leq n$ . Since  $\delta^k(cr) = c\delta^k(r)$  and  $\delta^k(rc) = \delta^k(r)c$  hold for all  $r \in R$  and  $c \in \text{Fix}(\delta)$ , we deduce that  $\bar{\delta}(cr) = c\bar{\delta}(r)$  and  $\bar{\delta}(rc) = \bar{\delta}(r)c$ .

(1) In view of  $t_{i,i} = 1$ , the  $(i, i)$  diagonal entry of  $\bar{\delta}(r)$  is

$$\bar{\delta}(r)_{i,i} = \frac{1}{n} \sum_{k=0}^{n-1} t_{i,i}^k \delta^k(r) = \frac{1}{n} \sum_{k=0}^{n-1} r = \frac{1}{n}(nr) = r.$$

For  $i \neq j$ , the  $(i, j)$  non-diagonal entry of  $\bar{\delta}(r)$  is

$$\bar{\delta}(r)_{i,j} = \frac{1}{n} \sum_{k=0}^{n-1} t_{j,i}^k \delta^k(r) = \frac{1}{n} \left( \sum_{k=0}^{n-1} t_{j,i}^k \right) r$$

and

$$\sum_{k=0}^{n-1} t_{j,i}^k = 1 + t_{j,i} + t_{j,i}^2 + \cdots + t_{j,i}^{n-1} = 0$$

follows from  $t_{j,i}^n = t_{j,1}^n t_{i,1}^{-n} = 1$  and

$$(1 - t_{j,i})(1 + t_{j,i} + t_{j,i}^2 + \cdots + t_{j,i}^{n-1}) = 1 - t_{j,i}^n = 0.$$

(2) The additive property of  $\bar{\delta}$  is clear. In order to prove the multiplicative property of  $\bar{\delta}$ , take  $r, s \in R$  and compute the  $(i, j)$  entry in the product of the  $n \times n$  matrices  $\bar{\delta}(r)$  and  $\bar{\delta}(s)$ :

$$\begin{aligned} (\bar{\delta}(r)\bar{\delta}(s))_{i,j} &= \frac{1}{n^2} \sum_{u=1}^n x_{i,u}(r)x_{u,j}(s) = \frac{1}{n^2} \sum_{u=1}^n \left( \sum_{k=0}^{n-1} t_{u,i}^k \delta^k(r) \right) \left( \sum_{l=0}^{n-1} t_{j,u}^l \delta^l(s) \right) \\ &= \frac{1}{n^2} \sum_{u=1}^n \left( \sum_{0 \leq k, l \leq n-1} t_{u,i}^k \delta^k(r) t_{j,u}^l \delta^l(s) \right) = \frac{1}{n^2} \sum_{0 \leq k, l \leq n-1} \left( \sum_{u=1}^n t_{j,u}^l t_{u,i}^k \right) \delta^k(r) \delta^l(s) \\ &\stackrel{(*)}{=} \frac{1}{n^2} \sum_{0 \leq q \leq n-1} \left( \sum_{u=1}^n t_{j,u}^q t_{u,i}^q \delta^q(r) \delta^q(s) \right) = \frac{1}{n^2} \sum_{0 \leq q \leq n-1} n t_{j,i}^q \delta^q(r) \delta^q(s) \\ &= \frac{1}{n} \sum_{q=0}^{n-1} t_{j,i}^q \delta^q(rs) = \frac{1}{n} x_{i,j}(rs) = (\bar{\delta}(rs))_{i,j}. \end{aligned}$$

The essential part of the above calculation is step (\*). We used that

$$\sum_{u=1}^n t_{j,u}^l t_{u,i}^k = \sum_{u=1}^n (t_{j,1} t_{u,1}^{-1})^l (t_{u,1} t_{i,1})^k = t_{j,1}^l \left( \sum_{u=1}^n t_{u,1}^{k-l} \right) t_{i,1}^k = 0$$

for any fixed pair  $(k, l)$  with  $0 \leq k, l \leq n-1$  and  $k \neq l$ . If  $\bar{\delta}(r) = 0_{n \times n}$ , then

$$0 = \sum_{i=1}^n x_{1,i}(r) = \sum_{i=1}^n \left( \sum_{k=0}^{n-1} t_{i,1}^k \delta^k(r) \right) = nr + \sum_{k=1}^{n-1} \left( \sum_{i=1}^n t_{i,1}^k \right) \delta^k(r) = nr,$$

whence  $r = \frac{1}{n}(nr) = 0$  follows. Thus  $\bar{\delta}$  is an embedding of rings.

(3) Now  $t_{j,i}^n = t_{j,1}^n t_{i,1}^{-n} = 1$  and  $\delta^n = \text{id}_R$  imply that  $t_{j,i}^n \delta^n(r) = r = t_{j,i}^0 \delta^0(r)$ . Since  $\delta(t_{j,i}^k \delta^k(r)) = t_{j,i}^k \delta^{k+1}(r)$  is a consequence of  $t_{j,i}^k = t_{j,1}^k t_{i,1}^{-k} \in \text{Fix}(\delta)$ , we have

$$\delta(x_{i,j}(r)) = \sum_{k=0}^{n-1} t_{j,i}^k \delta^{k+1}(r) = \sum_{l=1}^n t_{j,i}^{l-1} \delta^l(r) = t_{i,j} \sum_{l=1}^n t_{j,i}^l \delta^l(r) = t_{i,j} x_{i,j}(r)$$

proving that  $\bar{\delta}(r) \in M_n(R, \delta, T)$ .  $\square$

**3.3. Remark.** In the presence of  $t_{1,1}^n = t_{2,1}^n = \dots = t_{n,1}^n$  condition  $t_{1,1}^k + t_{2,1}^k + \dots + t_{n,1}^k = 0$  implies  $t_{1,1}^{k-n} + t_{2,1}^{k-n} + \dots + t_{n,1}^{k-n} = 0$ . Thus  $t_{1,1}^{-k} + t_{2,1}^{-k} + \dots + t_{n,1}^{-k} = 0$  in Theorem 3.2 is superfluous if  $t_{1,1}^n = t_{2,1}^n = \dots = t_{n,1}^n$ .

**3.4. Proposition.** Let  $R$  be an algebra over a field  $K$  of characteristic zero (notice that  $K \subseteq Z(R)$ ). If  $e \in K$  is a primitive  $n$ -th root of unity ( $e^n = 1 \neq e^k$  for all  $1 \leq k \leq n-1$ ), then for the  $n \times n$  transitive matrix  $P^{(e)} = [p_{i,j}]$  with  $p_{i,j} = e^{i-j}$ ,  $1 \leq i, j \leq n$  (see Example 2.4) we have  $p_{i,1}^n = 1$ ,  $p_{1,1}^k + p_{2,1}^k + \dots + p_{n,1}^k = 0$  and  $1 - p_{i,j}$  is a non-zero divisor (invertible) in  $K$  for all  $1 \leq i, j \leq n$  with  $i \neq j$ .

**Proof.** Now  $p_{i,1}^n = (e^{i-1})^n = (e^n)^{i-1} = 1$ . If  $1 \leq k \leq n-1$ , then  $1 - e^k \neq 0$  is invertible in  $K$  and

$$(1 - e^k)(1 + e^k + e^{2k} + \dots + e^{(n-1)k}) = 1 - e^{nk} = 0$$

implies that

$$p_{1,1}^k + p_{2,1}^k + \dots + p_{n,1}^k = 1 + e^k + e^{2k} + \dots + e^{(n-1)k} = 0.$$

If  $i \neq j$ , then  $1 - p_{i,j} = 1 - e^{i-j} \neq 0$  is invertible in  $K$ .  $\square$

**3.5. Corollary.** Let  $\frac{1}{2} \in R$  and  $\delta : R \rightarrow R$  be an arbitrary endomorphism. For  $r \in R$  the definition

$$\bar{\delta}(r) = \frac{1}{2} \begin{bmatrix} r + \delta(r) & r - \delta(r) \\ r - \delta(r) & r + \delta(r) \end{bmatrix}$$

gives an embedding  $\bar{\delta} : R \rightarrow M_2(R)$ . If  $\delta^2 = \text{id}_R$ , then  $\bar{\delta}$  is an  $R \rightarrow M_2(R, \delta, P)$  embedding (for  $P$  see 2.4).

**Proof.** Take  $n = 2$  and  $t_{1,1} = t_{2,2} = 1$ ,  $t_{1,2} = t_{2,1} = -1$  in Theorem 3.2.  $\square$

#### 4. THE RIGHT AND LEFT DETERMINANTS OF A SUPERMATRIX

The following definitions and the basic results about the symmetric and Lie-nilpotent analogues of the classical determinant theory can be found in [Do, S1, S3, SvW].

Let  $S_n$  denote the symmetric group of all permutations of the set  $\{1, 2, \dots, n\}$ . For an  $n \times n$  matrix  $A = [a_{i,j}]$  over an arbitrary (possibly non-commutative) ring or algebra  $R$  with 1, the element

$$\begin{aligned} \text{sdet}(A) &= \sum_{\tau, \pi \in S_n} \text{sgn}(\pi) a_{\tau(1), \pi(\tau(1))} \cdots a_{\tau(t), \pi(\tau(t))} \cdots a_{\tau(n), \pi(\tau(n))} \\ &= \sum_{\alpha, \beta \in S_n} \text{sgn}(\alpha) \text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(t), \beta(t)} \cdots a_{\alpha(n), \beta(n)} \end{aligned}$$

of  $R$  is the *symmetric determinant* of  $A$ . The *preadjoint matrix*  $A^* = [a_{r,s}^*]$  of  $A = [a_{i,j}]$  is defined as the following natural symmetrization of the classical adjoint:

$$\begin{aligned} a_{r,s}^* &= \sum_{\tau, \pi} \text{sgn}(\pi) a_{\tau(1), \pi(\tau(1))} \cdots a_{\tau(s-1), \pi(\tau(s-1))} a_{\tau(s+1), \pi(\tau(s+1))} \cdots a_{\tau(n), \pi(\tau(n))} \\ &= \sum_{\alpha, \beta} \text{sgn}(\alpha) \text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(s-1), \beta(s-1)} a_{\alpha(s+1), \beta(s+1)} \cdots a_{\alpha(n), \beta(n)}, \end{aligned}$$

where the first sum is taken over all  $\tau, \pi \in S_n$  with  $\tau(s) = s$  and  $\pi(s) = r$  (while the second sum is taken over all  $\alpha, \beta \in S_n$  with  $\alpha(s) = s$  and  $\beta(s) = r$ ). We note that

the  $(r, s)$  entry of  $A^*$  is exactly the signed symmetric determinant  $(-1)^{r+s} \text{sdet}(A_{s,r})$  of the  $(n-1) \times (n-1)$  minor  $A_{s,r}$  of  $A$  arising from the deletion of the  $s$ -th row and the  $r$ -th column of  $A$ . If  $R$  is commutative, then  $\text{sdet}(A) = n! \det(A)$  and  $A^* = (n-1)! \text{adj}(A)$ , where  $\det(A)$  and  $\text{adj}(A)$  denote the ordinary determinant and adjoint of  $A$ .

The right adjoint sequence  $(P_k)_{k \geq 1}$  of  $A$  is defined by the recursion:  $P_1 = A^*$  and  $P_{k+1} = (AP_1 \cdots P_k)^*$  for  $k \geq 1$ . The  $k$ -th right determinant is the trace of  $AP_1 \cdots P_k$ :

$$\text{rdet}_{(k)}(A) = \text{tr}(AP_1 \cdots P_k).$$

The left adjoint sequence  $(Q_k)_{k \geq 1}$  can be defined analogously:  $Q_1 = A^*$  and  $Q_{k+1} = (Q_k \cdots Q_1 A)^*$  for  $k \geq 1$ . The  $k$ -th left determinant of  $A$  is

$$\text{ldet}_{(k)}(A) = \text{tr}(Q_k \cdots Q_1 A).$$

Clearly,  $\text{rdet}_{(k+1)}(A) = \text{rdet}_{(k)}(AA^*)$  and  $\text{ldet}_{(k+1)}(A) = \text{ldet}_{(k)}(A^*A)$ . We note that

$$\text{rdet}_{(1)}(A) = \text{tr}(AA^*) = \text{sdet}(A) = \text{tr}(A^*A) = \text{ldet}_{(1)}(A).$$

As we can see in Section 5, the following theorem is a broad generalization of one of the main results in [S2].

**4.1. Theorem.** *Let  $\delta : R \rightarrow R$  be an endomorphism and  $T = [t_{i,j}]$  be a transitive matrix in  $M_n(Z(R))$ . If  $A \in M_n(R, \delta, T)$  is a supermatrix, then  $A^* \in M_n(R, \delta, T)$ . In other words, the supermatrix algebra  $M_n(R, \delta, T)$  is closed with respect to taking the preadjoint.*

**Proof.** The  $(r, s)$  entry of  $A^*$  is

$$a_{r,s}^* = \sum_{\tau, \pi} \text{sgn}(\pi) a_{\tau(1), \pi(\tau(1))} \cdots a_{\tau(s-1), \pi(\tau(s-1))} a_{\tau(s+1), \pi(\tau(s+1))} \cdots a_{\tau(n), \pi(\tau(n))},$$

where  $A = [a_{i,j}]$  and the sum is taken over all  $\tau, \pi \in S_n$  with  $\tau(s) = s$  and  $\pi(s) = r$ . In order to see that  $A^* \in M_n(R, \delta, T)$ , we prove that  $\delta(a_{r,s}^*) = t_{r,s} a_{r,s}^*$  for all  $1 \leq r, s \leq n$ . Using  $\delta(a_{\tau(i), \pi(\tau(i))}) = t_{\tau(i), \pi(\tau(i))} a_{\tau(i), \pi(\tau(i))}$  we obtain that

$$\delta(a_{r,s}^*) = \sum_{\tau, \pi} \text{sgn}(\pi) b(1, \tau, \pi) \cdots b(s-1, \tau, \pi) b(s+1, \tau, \pi) \cdots b(n, \tau, \pi),$$

where  $b(i, \tau, \pi) = t_{\tau(i), \pi(\tau(i))} a_{\tau(i), \pi(\tau(i))}$ . Since  $t_{\tau(i), \pi(\tau(i))} \in Z(R)$  and  $\tau(i) \in \{1, \dots, s-1, s+1, \dots, n\}$  for all  $i \in \{1, \dots, s-1, s+1, \dots, n\}$ , we have

$$\begin{aligned} & t_{\tau(1), \pi(\tau(1))} \cdots t_{\tau(s-1), \pi(\tau(s-1))} t_{\tau(s+1), \pi(\tau(s+1))} \cdots t_{\tau(n), \pi(\tau(n))} \\ &= t_{1, \pi(1)} \cdots t_{s-1, \pi(s-1)} t_{s+1, \pi(s+1)} \cdots t_{n, \pi(n)}. \end{aligned}$$

The product  $t_{1, \pi(1)} \cdots t_{s-1, \pi(s-1)} t_{s+1, \pi(s+1)} \cdots t_{n, \pi(n)}$  can be rearranged according to the cycles of  $\pi$ . For each cycle  $(i, \pi(i), \dots, \pi^k(i))$  of the permutation  $\pi$  (of length  $k+1$  say) not containing  $s$  (and hence  $r$ ) we have a factor ("sub-product")  $t_{i, \pi(i)} t_{\pi(i), \pi^2(i)} \cdots t_{\pi^k(i), \pi^{k+1}(i)}$  of the above product and the transitivity of  $T$  gives that

$$t_{i, \pi(i)} t_{\pi(i), \pi^2(i)} \cdots t_{\pi^k(i), \pi^{k+1}(i)} = t_{i,i} = 1.$$

The only cycle of  $\pi$  containing  $s$  (as well as  $r$ ) is of the form  $(r, \pi(r), \dots, \pi^l(r) = s)$  for some  $l \geq 1$ . The corresponding factor ("sub-product") of

$$t_{1, \pi(1)} \cdots t_{s-1, \pi(s-1)} t_{s+1, \pi(s+1)} \cdots t_{n, \pi(n)}$$

does not contain  $t_{\pi^l(r), \pi^{l+1}(r)} = t_{s, \pi(s)} = t_{s, r}$  and the transitivity of  $T$  gives that

$$t_{r, \pi(r)} t_{\pi(r), \pi^2(r)} \cdots t_{\pi^{l-1}(r), \pi^l(r)} = t_{r, s}.$$

It follows that

$$t_{\tau(1), \pi(\tau(1))} \cdots t_{\tau(s-1), \pi(\tau(s-1))} t_{\tau(s+1), \pi(\tau(s+1))} \cdots t_{\tau(n), \pi(\tau(n))} = t_{r, s}$$

for all  $\tau, \pi \in S_n$  with  $\tau(s) = s$  and  $\pi(s) = r$ . Thus

$$b(1, \tau, \pi) \cdots b(s-1, \tau, \pi) b(s+1, \tau, \pi) \cdots b(n, \tau, \pi)$$

$$= t_{r, s} a_{\tau(1), \pi(\tau(1))} \cdots a_{\tau(s-1), \pi(\tau(s-1))} a_{\tau(s+1), \pi(\tau(s+1))} \cdots a_{\tau(n), \pi(\tau(n))},$$

whence  $\delta(a_{r, s}^*) =$

$$t_{r, s} \sum_{\tau, \pi} \text{sgn}(\pi) a_{\tau(1), \pi(\tau(1))} \cdots a_{\tau(s-1), \pi(\tau(s-1))} a_{\tau(s+1), \pi(\tau(s+1))} \cdots a_{\tau(n), \pi(\tau(n))} = t_{r, s} a_{r, s}^*$$

follows.  $\square$

**4.2. Corollary.** *Let  $\delta : R \rightarrow R$  be an endomorphism and  $T = [t_{i,j}]$  be a transitive matrix in  $M_n(Z(R))$ . If  $A \in M_n(R, \delta, T)$  is a supermatrix, then we have*

$$\text{rdet}_{(k)}(A), \text{ldet}_{(k)}(A) \in \text{Fix}(\delta)$$

for all  $k \geq 1$ . In particular  $\text{sdet}(A) = \text{rdet}_{(1)}(A) = \text{ldet}_{(1)}(A) \in \text{Fix}(\delta)$ .

**Proof.** The repeated application of Theorem 4.1 gives that the recursion  $P_1 = A^*$  and  $P_{k+1} = (AP_1 \cdots P_k)^*$  starting from a supermatrix  $A \in M_n(R, \delta, T)$  defines a sequence  $(P_k)_{k \geq 1}$  in  $M_n(R, \delta, T)$ . Since  $\text{rdet}_{(k)}(A) = \text{tr}(AP_1 \cdots P_k)$  is the sum of the diagonal entries of the product supermatrix  $AP_1 \cdots P_k \in M_n(R, \delta, T)$  and each diagonal entry of a supermatrix (in  $M_n(R, \delta, T)$ ) is in  $\text{Fix}(\delta)$ , the proof is complete. The poof of  $\text{ldet}_{(k)}(A) \in \text{Fix}(\delta)$  is similar.  $\square$

Let  $R[z]$  denote the ring of polynomials of the single commuting indeterminate  $z$ , with coefficients in  $R$ . The  $k$ -th right (left) characteristic polynomial of  $A$  is the  $k$ -th right (left) determinant of the  $n \times n$  matrix  $zI_n - A$  in  $M_n(R[z])$ :

$$p_{A,k}(z) = \text{rdet}_{(k)}(zI_n - A) \text{ and } q_{A,k}(z) = \text{ldet}_{(k)}(zI_n - A).$$

Notice that  $p_{A,k}(z)$  is of the following form:

$$p_{A,k}(z) = \lambda_0^{(k)} + \lambda_1^{(k)} z + \cdots + \lambda_{n^k-1}^{(k)} z^{n^k-1} + \lambda_{n^k}^{(k)} z^{n^k},$$

where  $\lambda_0^{(k)}, \lambda_1^{(k)}, \dots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$  and  $\lambda_{n^k}^{(k)} = n \{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$ .

**4.3. Corollary.** *Let  $\delta : R \rightarrow R$  be an endomorphism and  $T = [t_{i,j}]$  be a transitive matrix in  $M_n(Z(R))$ . If  $A \in M_n(R, \delta, T)$  is a supermatrix, then we have*

$$p_{A,k}(z), q_{A,k}(z) \in \text{Fix}(\delta)[z]$$

for all  $k \geq 1$ . In other words, the coefficients of the right  $p_{A,k}(z) = \text{rdet}_{(k)}(zI_n - A)$  and left  $q_{A,k}(z) = \text{ldet}_{(k)}(zI_n - A)$  characteristic polynomials are in  $\text{Fix}(\delta)$ .

**Proof.** Now  $\delta : R \rightarrow R$  can be extended to an endomorphism  $\delta_z : R[z] \rightarrow R[z]$  of the polynomial ring (algebra): for  $r_0, r_1, \dots, r_m \in R$  take

$$\delta_z(r_0 + r_1 z + \cdots + r_m z^m) = \delta(r_0) + \delta(r_1) z + \cdots + \delta(r_m) z^m.$$



Since  $T$  can be considered as a transitive matrix over  $Z(R[z]) = Z(R)[z]$  and  $zI_n - A \in M_n(R[z], \delta_z, T)$ , Corollary 4.2 gives that  $p_{A,k}(z) = \text{rdet}_{(k)}(zI_n - A)$  and  $q_{A,k}(z) = \text{ldet}_{(k)}(zI_n - A)$  are in  $\text{Fix}(\delta_z) = \text{Fix}(\delta)[z]$ .  $\square$

**4.4. Theorem.** *Let  $\delta : R \rightarrow R$  be an endomorphism and  $T = [t_{i,j}]$  be a transitive matrix in  $M_n(Z(R))$ . If  $R$  satisfies the polynomial identity*

$$[[[\dots [x_1, x_2], x_3], \dots], x_k], x_{k+1}] = 0$$

*( $R$  is Lie nilpotent of index  $k$ ) and  $A \in M_n(R, \delta, T)$  is a supermatrix, then a right Cayley-Hamilton identity*

$$(A)p_{A,k} = I_n \lambda_0^{(k)} + A \lambda_1^{(k)} + \dots + A^{n^k-1} \lambda_{n^k-1}^{(k)} + A^{n^k} \lambda_{n^k}^{(k)} = 0$$

*holds, where the coefficients  $\lambda_i^{(k)}$ ,  $0 \leq i \leq n^k$  of  $p_{A,k}(z) = \text{rdet}_{(k)}(zI_n - A)$  are in  $\text{Fix}(\delta)$ . If  $\lambda_{n^k}^{(k)} = n \{(n-1)!\}^{1+n+n^2+\dots+n^{k-1}}$  is invertible in  $R$  and  $\text{Fix}(\delta) \subseteq Z(R)$ , then the above identity provides the integrality of  $M_n(R, \delta, T)$  over  $Z(R)$  (of degree  $n^k$ ).*

**Proof.** Since one of the main results of [S1] is that

$$(A)p_{A,k} = I_n \lambda_0^{(k)} + A \lambda_1^{(k)} + \dots + A^{n^k-1} \lambda_{n^k-1}^{(k)} + A^{n^k} \lambda_{n^k}^{(k)} = 0$$

holds for  $A \in M_n(R)$ , Corollary 4.3 can be used.  $\square$

The combination of Theorems 3.2 and 4.4 yields the following.

**4.5. Theorem.** *Let  $R$  be a Lie nilpotent algebra of index  $k \geq 1$  over a field  $K$  of characteristic zero and let  $e \in K$  be a primitive  $n$ -th root of unity. If  $\delta : R \rightarrow R$  is a  $K$ -automorphism ( $K \subseteq \text{Fix}(\delta)$ ) with  $\delta^n = \text{id}_R$ , then  $R$  is right (and left) integral over  $\text{Fix}(\delta)$  of degree  $n^k$ . In other words, for any  $r \in R$  we have*

$$c'_0 + r c'_1 + \dots + r^{n^k-1} c'_{n^k-1} + r^{n^k} = 0 = c''_0 + c''_1 r + \dots + c''_{n^k-1} r^{n^k-1} + r^{n^k}$$

*for some  $c'_i, c''_i \in \text{Fix}(\delta)$ ,  $0 \leq i \leq n^k - 1$ .*

**Proof.** Let  $P^{(e)} = [p_{i,j}]$  be the same  $n \times n$  transitive matrix (with  $p_{i,j} = e^{i-j}$ ,  $1 \leq i, j \leq n$ ) as in Proposition 3.4. The application of Theorem 3.2 provides an embedding  $\bar{\delta} : R \rightarrow M_n(R, \delta, P^{(e)})$  such that  $\bar{\delta}(cr) = c\bar{\delta}(r)$  and  $\bar{\delta}(rc) = \bar{\delta}(r)c$  for all  $r \in R$  and  $c \in \text{Fix}(\delta)$ . Theorem 4.4 ensures that

$$I_n \lambda_0^{(k)} + \bar{\delta}(r) \lambda_1^{(k)} + \dots + (\bar{\delta}(r))^{n^k-1} \lambda_{n^k-1}^{(k)} + (\bar{\delta}(r))^{n^k} \lambda_{n^k}^{(k)} = 0$$

holds, where the coefficients of the  $k$ -th right characteristic polynomial

$$p_{\bar{\delta}(r),k}(z) = \text{rdet}_{(k)}(zI_n - \bar{\delta}(r)) = \lambda_0^{(k)} + \lambda_1^{(k)} z + \dots + \lambda_{n^k-1}^{(k)} z^{n^k-1} + \lambda_{n^k}^{(k)} z^{n^k}$$

of  $\bar{\delta}(r) \in M_n(R, \delta, P^{(e)})$  are in  $\text{Fix}(\delta)$ . Since  $\lambda_{n^k}^{(k)} = n \{(n-1)!\}^{1+n+n^2+\dots+n^{k-1}}$  is invertible in  $K$ , for  $c'_i = \lambda_i^{(k)} \cdot (\lambda_{n^k}^{(k)})^{-1} \in \text{Fix}(\delta)$ ,  $0 \leq i \leq n^k - 1$  we have

$$\bar{\delta}(c'_0 + r c'_1 + \dots + r^{n^k-1} c'_{n^k-1} + r^{n^k}) = I_n c'_0 + \bar{\delta}(r) c'_1 + \dots + (\bar{\delta}(r))^{n^k-1} c'_{n^k-1} + (\bar{\delta}(r))^{n^k} = 0.$$

Thus  $\ker(\bar{\delta}) = \{0\}$  gives the desired right integrality (the case of left integrality is similar).  $\square$

## 5. SUPERMATRIX ALGEBRAS OVER THE GRASSMANN ALGEBRA

A  $\mathbb{Z}_2$ -grading of a ring  $R$  is a pair  $(R_0, R_1)$ , where  $R_0$  and  $R_1$  are additive subgroups of  $R$  such that  $R = R_0 \oplus R_1$  is a direct sum and  $R_i R_j \subseteq R_{i+j}$  for all  $i, j \in \{0, 1\}$  and  $i + j$  is taken modulo 2. The relation  $R_0 R_0 \subseteq R_0$  ensures that  $R_0$  is a subring of  $R$ . Now any element  $r \in R$  can be uniquely written as  $r = r_0 + r_1$ , where  $r_0 \in R_0$  and  $r_1 \in R_1$ . It is easy to see that the existence of  $1 \in R$  implies that  $1 \in R_0$ . The function  $\rho : R \rightarrow R$  defined by  $\rho(r_0 + r_1) = r_0 - r_1$  is an automorphism of  $R$  with  $\rho^2 = \text{id}_R$ .

**5.1. Example.** A certain supermatrix algebra  $M_{n,d}(R)$  is considered in [S2]. In view of  $\text{Fix}(\rho) = R_0$  and  $\rho(r_0 + r_1) = -(r_0 + r_1) \iff r_0 = 0$ , it is straightforward to see that  $M_{n,d}(R) = M_n(R, \rho, P(d, n))$ , where  $P(d, n)$  is the  $n \times n$  blow up matrix in 2.4 with  $m = n$ .

The Grassmann (exterior) algebra

$$E = K \langle v_1, v_2, \dots, v_i, \dots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i < j \rangle$$

over a field  $K$  (of characteristic zero) generated by the infinite sequence of anticommutative indeterminates  $(v_i)_{i \geq 1}$  is a typical example of a  $\mathbb{Z}_2$ -graded algebra. Using the well known  $\mathbb{Z}_2$ -grading  $E = E_0 \oplus E_1$  and the corresponding automorphism  $\varepsilon : E \rightarrow E$ , we obtain the classical supermatrix algebra  $M_{n,d}(E) = M_n(E, \varepsilon, P(d, n))$ .

Since  $\varepsilon^2 = \text{id}_E$  implies that  $(\varepsilon_n)^2 = \text{id}_{M_n(E)}$ , we can take  $R = M_n(E)$  and  $\delta = \varepsilon_n$  in Corollary 3.5. Thus we obtain the well known embedding

$$(\overline{\varepsilon_n}) : M_n(E) \rightarrow M_2(M_n(E), \varepsilon_n, P) \cong M_{2n}(E, \varepsilon, P(n, 2n)) = M_{2n,n}(E)$$

of  $M_n(E)$  into the supermatrix algebra  $M_{2n,n}(E)$ .

In view of  $\text{Fix}(\varepsilon) = E_0 = Z(E)$ , the application of Theorem 4.4 gives that  $M_{n,d}(E)$  is integral over  $E_0$  of degree  $n^2$  (see Theorem 3.3 in [S2]). Thus the main results of [S2] directly follow from 4.1, 4.2, 4.3 and 4.4.  $\square$

We note that the T-ideal of the polynomial identities (with coefficients in  $K$ ) satisfied by  $M_{n,d}(E)$  plays an important role in Kemer's classification of the T-prime T-ideals (see [K]).

**5.2. Example.** For an integer  $k \geq 0$  let

$$E(k) = \bigoplus_{1 \leq i_1 < \dots < i_k} K v_{i_1} \cdots v_{i_k}$$

denote the  $k$ -homogeneous component of  $E$ , i.e. the  $K$ -linear span of all products  $v_{i_1} \cdots v_{i_k}$  of length  $k$  ( $E(0) = K$ ). If  $e \in K$  is a primitive  $n$ -th root of unity, then define an automorphism  $\rho_e : E \rightarrow E$  as follows

$$\rho_e(h(v_1, v_2, \dots, v_k, \dots)) = h(ev_1, ev_2, \dots, ev_k, \dots),$$

where each generator  $v_k$  in  $h \in E$  is replaced by  $ev_k$ .

Consider the supermatrix algebra  $M_n(E, \rho_e, P^{(e)})$ , where  $P^{(e)} = [p_{i,j}]$  is the same  $n \times n$  transitive matrix (with  $p_{i,j} = e^{i-j}$ ,  $1 \leq i, j \leq n$ ) as in Proposition 3.4. If  $A = [h_{i,j}]$  is a supermatrix in  $M_n(E, \rho_e, P^{(e)})$ , then we have  $\rho_e(h_{i,j}) = e^{i-j} h_{i,j}$  for all  $1 \leq i, j \leq n$ . Since for an integer  $0 \leq m \leq n-1$

$$(ev_{i_1}) \cdots (ev_{i_k}) = e^k v_{i_1} \cdots v_{i_k} = e^m v_{i_1} \cdots v_{i_k}$$

holds if and only if  $k - m$  is divisible by  $n$ , we obtain that

$$E_{m,n} = \{h \in E \mid \rho_e(h) = e^m h\} = \bigoplus_{u=0}^{\infty} E(m + nu)$$

and

$$E_{-m,n} = \{h \in E \mid \rho_e(h) = e^{-m} h\} = E_{n-m,n}.$$

Thus the shape of  $M_n(E, \rho_e, P^{(e)})$  is the following:

$$M_n(E, \rho_e, P^{(e)}) = \begin{bmatrix} E_{0,n} & E_{-1,n} & \cdots & E_{-(n-2),n} & E_{-(n-1),n} \\ E_{1,n} & E_{0,n} & E_{-1,n} & \ddots & E_{-(n-2),n} \\ \vdots & E_{1,n} & \ddots & \ddots & \vdots \\ E_{n-2,n} & \ddots & \ddots & E_{0,n} & E_{-1,n} \\ E_{n-1,n} & E_{n-2,n} & \cdots & E_{1,n} & E_{0,n} \end{bmatrix}.$$

Since  $(\rho_e)^n = \text{id}_E$ , Theorem 3.2 provides an embedding  $\overline{\rho}_e : E \longrightarrow M_n(E, \rho_e, P^{(e)})$ . In view of the Lie nilpotency of  $E$  (of index 2), the application of Theorem 4.4 gives that any matrix  $A \in M_n(E, \rho_e, P^{(e)})$  satisfies a right Cayley-Hamilton identity of degree  $n^2$  with (right) coefficients from  $\text{Fix}(\rho_e) = E_{0,n}$ . If  $n$  is even, then  $E_{0,n} \subseteq E_0 = Z(E)$  is a central subalgebra and the coefficients in the mentioned (right-left) Cayley-Hamilton identity are central. The application of Theorem 4.5 gives that  $E$  is right (and left) integral over  $E_{0,n}$  of degree  $n^2$ .

If  $n = 2$  and  $e = -1$ , then  $E_{0,2} = E_0$  is the even and  $E_{1,2} = E_1$  is the odd part of the Grassmann algebra  $E$  and  $M_2(E, \rho_{-1}, P^{(-1)}) = M_2(E, \varepsilon, P)$  (for  $P$  see 2.4).  $\square$

**5.3. Example.** For  $g \in E$ , let  $\sigma : E \longrightarrow E$  be the following map:

$$\sigma(g) = (1 + v_1)g(1 - v_1).$$

Clearly,  $1 - v_1 = (1 + v_1)^{-1}$  implies that  $\sigma$  is a conjugate automorphism of  $E$ . The blow up  $Q(d, n)$  of the transitive matrix

$$Q = \begin{bmatrix} 1 & 1 + v_1 v_2 \\ 1 - v_1 v_2 & 1 \end{bmatrix}$$

is in  $M_n(E_0)$  (notice that  $E_0 = Z(E)$ ). Thus we can form the supermatrix algebra  $M_n(E, \sigma, Q(d, n))$ . The block structure of a supermatrix  $A \in M_n(E, \sigma, Q(d, n))$  is the following

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where the square blocks  $A_{1,1}$  and  $A_{2,2}$  are of sizes  $d \times d$  and  $(m - d) \times (m - d)$  and the rectangular blocks  $A_{1,2}$  and  $A_{2,1}$  are of sizes  $d \times (m - d)$  and  $(m - d) \times d$ . The entries of  $A_{1,1}$  and  $A_{2,2}$  are in

$$\text{Fix}(\sigma) = \{g \in E \mid (1 + v_1)g(1 - v_1) = g\} = \{g \in E \mid v_1 g - g v_1 = 0\} = \text{Cen}(v_1) = E_0 + E_0 v_1,$$

where  $\text{Cen}(v_1)$  denotes the centralizer of  $v_1$ . The entries of  $A_{1,2}$  are in

$$\begin{aligned} \Omega_{1,2} &= \{g \in E \mid (1 + v_1)g(1 - v_1) = (1 + v_1 v_2)g\} = \{g \in E \mid v_1 g - g v_1 = v_1 v_2 g\} \\ &= \{g_0 + g_1 \mid g_0 \in E_0, g_1 \in E_1, v_1 g_1 - g_1 v_1 = v_1 v_2 g_0 \text{ and } v_1 v_2 g_1 = 0\} \\ &= \{g_0 + g_1 \mid g_0 \in E_0, 2g_1 - v_2 g_0 \in E_0 v_1\} \subseteq E_0 + E_0 v_1 + E_0 v_2. \end{aligned}$$

The entries of  $A_{2,1}$  are in

$$\begin{aligned}\Omega_{2,1} &= \{g \in E \mid (1 + v_1)g(1 - v_1) = (1 - v_1v_2)g\} = \{g \in E \mid v_1g - gv_1 = -v_1v_2g\} \\ &= \{g_0 + g_1 \mid g_0 \in E_0, g_1 \in E_1, v_1g_1 - g_1v_1 = -v_1v_2g_0 \text{ and } v_1v_2g_1 = 0\} \\ &= \{g_0 + g_1 \mid g_0 \in E_0, 2g_1 + v_2g_0 \in E_0v_1\} \subseteq E_0 + E_0v_1 + E_0v_2\end{aligned}$$

As a consequence, we obtain that the shape of  $M_n(E, \sigma, Q(d, n))$  is the following:

$$M_n(E, \sigma, Q(d, n)) = \begin{bmatrix} E_0 + E_0v_1 & \Omega_{1,2} \\ \Omega_{2,1} & E_0 + E_0v_1 \end{bmatrix}$$

with diagonal blocks of sizes  $d \times d$  and  $(m - d) \times (m - d)$ . In view of the Lie nilpotency of  $E$  (of index 2), the application of Theorem 4.4 gives that any matrix  $A \in M_n(E, \sigma, Q(d, n))$  satisfies a right Cayley-Hamilton identity of degree  $n^2$  with (right) coefficients from  $E_0 + E_0v_1$ .  $\square$

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